Turbulence in Magnetised Plasmas

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Turbulence in Magnetised Plasmas

- nonlinearity of small disturbances on an equilibrium

  † three wave interactions

  † energy transfer, cascading

- incompressible turbulence models

  † simple fluid turbulence, role of pressure to maintain incompressibility

  † cascades of energy, vorticity ("enstrophy"), role of vortex tubes in 2D and 3D models

  † 2D MHD turbulence, role of magnetic field to maintain incompressibility

- simple drift effects in a magnetised plasma with gradients

  † dissipative coupling, effect on cascades

  † evolution of spectra, physical meaning of cascades

  † varying properties of nonlinear couplings

- the transport problem

Various Nonlinear Effects

- rapid space/time variation of parameters (e.g., shocks, isolated jets)
- quasilinear interaction between small waves to alter the background

  † each wave ($\mathbf{k}$) beats against itself ($\mathbf{k}'$)

  † background is wavenumber zero

$$\mathbf{k} + \mathbf{k}' = 0$$

- turbulence — incoherent interaction with many wave combinations

  † each wave ($\mathbf{k}$) is forced upon by two other beat waves ($\mathbf{k}'$ and $\mathbf{k}''$)

  † many distinct pairs $\{\mathbf{k}', \mathbf{k}''\}$ with no relation to $\mathbf{k}$

$$\mathbf{k} + \mathbf{k}' + \mathbf{k}'' = 0$$

many degrees of freedom, incoherent, statistical

Small Disturbances on an Equilibrium

- ordering in general — gradients multiplied by constant parameters

$$\Delta_\perp \ll L_\perp \qquad \Longrightarrow \qquad (p_e + \widetilde{p}_e)\,\nabla\cdot\mathbf{v} \quad\rightarrow\quad p_e\nabla\cdot\mathbf{v}$$

- background may be inhomogeneous (define x as down-gradient)

$$\nabla p_e \rightarrow -\frac{p_e}{L_\perp}\nabla x \qquad \text{where} \qquad \nabla x = -L_\perp\nabla\log p_e$$

- "mixing level" disturbances

$$\nabla\widetilde{p}_e \sim \nabla p_e \;\Longrightarrow\; \frac{\widetilde{p}_e}{p_e} \sim \frac{\Delta_\perp}{L_\perp} \ll 1$$

- nonlinearity remains in advection effects — a nonlinear term and a linear forcing term

$$\mathbf{v}_E\cdot\nabla\,(p_e + \widetilde{p}_e) = \frac{c}{B}\mathbf{b}\cdot\nabla\widetilde{\phi}\times\nabla\widetilde{p}_e - \frac{p_e}{L_\perp}\,v_E^x \qquad \text{where} \qquad v_E^x = \frac{c}{B}\mathbf{b}\cdot\nabla\widetilde{\phi}\times\nabla x$$

keep nonlinearities where quadratic under gradients

Incompressible Hydrodynamics

- start with MHD, neglect magnetic field

$$\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\frac{1}{\rho}\,\nabla p$$

- take curl, treat ρ as constant, neglect $\nabla \cdot \mathbf{v}$

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) \nabla\times\mathbf{v} = (\nabla\times\mathbf{v}) \cdot \nabla\mathbf{v}$$

- pressure submerges — only role is to maintain incompressibility

$$\text{let} \quad \frac{\partial}{\partial t}\nabla \cdot \mathbf{v} = 0 \qquad\qquad \text{then} \qquad \nabla^2 p = -\nabla \cdot (\rho\mathbf{v} \cdot \nabla\mathbf{v})$$

- leads to "projection methods" for computations

The Cascade to Smaller Scales

- the "eddy mitosis" model: vortices sheared apart, into smaller ones about half the size
- assume: energy input ("stirring") and loss ("dissipation") occur in well separated ranges in scale
  ◦ situation of "high Reynolds number" meaning turbulent mixing $\gg$ viscous or collisional diffusion
- at scale n, have kinetic energy, $E_n = v_n^2/2$, and "eddy turnover time" inverse to vorticity, $(kv)_n$
- during the mitosis process, energy is conserved $\rightarrow$ power law

$$(kv)_{n-1}\, E_{n-1} = (kv)_n\, E_n \qquad\qquad k_n = 2k_{n-1}$$

- in this "inertial range" one finds the Kolmogorov scaling law

$$(E_n/k_n) \propto k_n^{-5/3} \qquad\qquad \text{density of states} \qquad k_n$$

- the vorticity *increases* towards smaller scales $\rightarrow$ enstrophy is *produced*

$$(kv)_n \propto k_n^{2/3}$$

Enstrophy in Incompressible Hydrodynamics

- Euler equation in 3D

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\nabla\times\mathbf{v} = (\nabla\times\mathbf{v})\cdot\nabla\mathbf{v}$$

- note mean squared vorticity ("enstrophy") is not generally conserved

$$\frac{\partial W}{\partial t} + \nabla\cdot(W\,\mathbf{v}) = [(\nabla\times\mathbf{v})(\nabla\times\mathbf{v})]:[\nabla\mathbf{v}] \qquad \text{where} \qquad W = \frac{1}{2}(\nabla\times\mathbf{v})\cdot(\nabla\times\mathbf{v})$$

- enstrophy is transported by the velocity, but grows if . . .

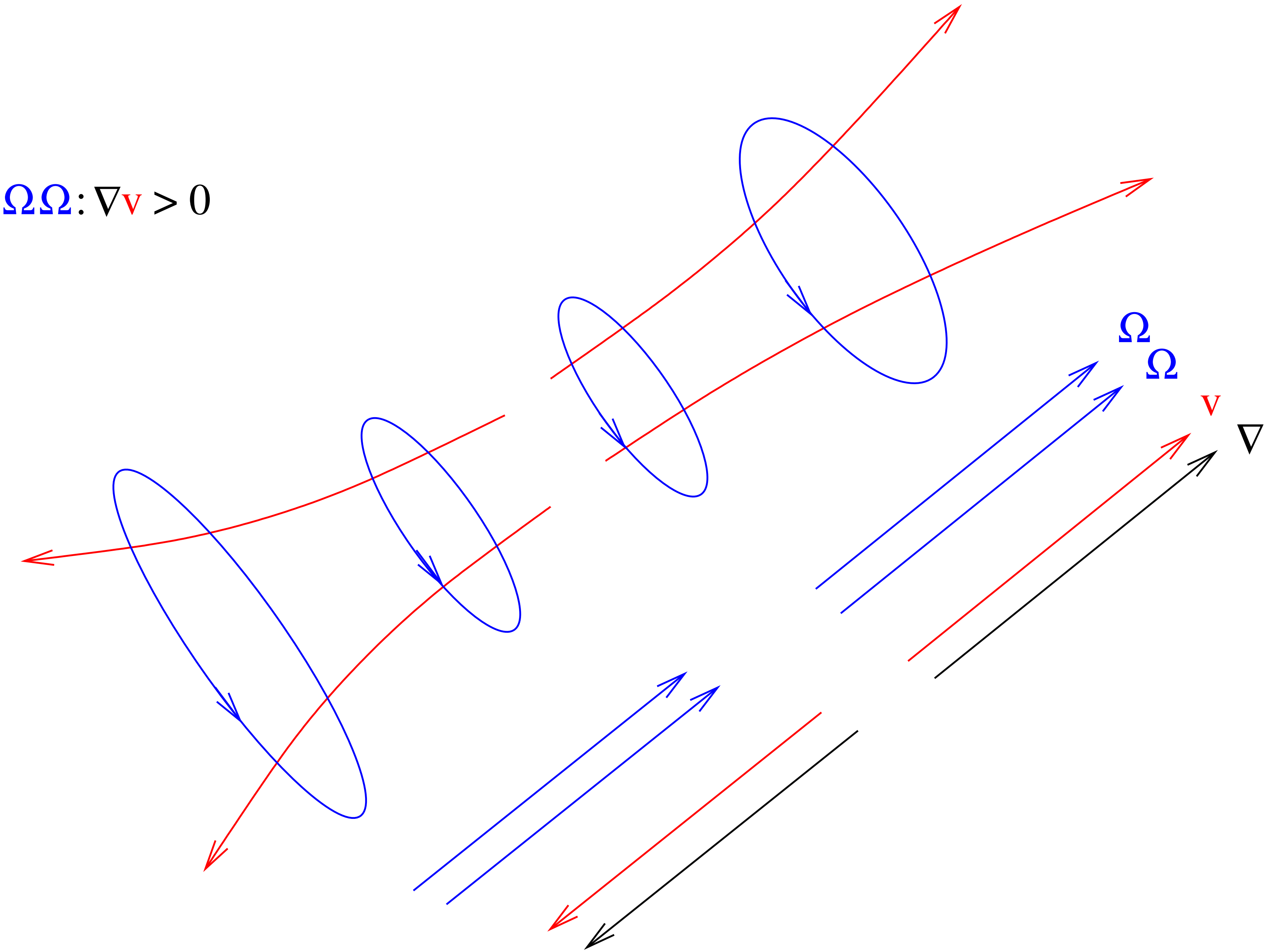
† the velocity has a component along the vorticity, and also diverges in that direction

vortex tube stretching in 3D

Vortex Tube Stretching

type of motion necessary to entrophy production

$\Omega\Omega : \nabla v > 0$



What You Can Learn Just From Equations

- energy conservation, energy transfer to smaller scales
  - statistical redistribution, with more states available at smaller scale
  - enstrophy must increase

- geometry: enstrophy increase is described by a definite quantity
  - this quantity can only be positive if there are vortex tubes which are stretched by the flow

Kolmogorov cascade process must proceed through vortex tube stretching

- the above is found merely by examining the properties of the equations
  - actually solving them was not necessary

2D Incompressible Hydrodynamics

- in 2D one must have $\nabla\times\mathbf{v} \perp \mathbf{v}$ ... let $\hat{\mathbf{s}}$ be the normal to the plane

$$\nabla\cdot\mathbf{v} = 0 \qquad \Longrightarrow \qquad \mathbf{v} = \hat{\mathbf{s}}\times\nabla\psi \qquad \Longrightarrow \qquad (\nabla\times\mathbf{v}) = \hat{\mathbf{s}}\nabla_\perp^2\psi$$

- find the 2D Euler equation

$$\frac{\partial\Omega}{\partial t} + \mathbf{v}\cdot\nabla\Omega = 0 \qquad \text{with} \qquad \Omega = \nabla_\perp^2\psi \qquad \text{and} \qquad \mathbf{v} = \hat{\mathbf{s}}\times\nabla\psi$$

- hence the enstrophy (W) is conserved, along with the energy (U)

$$\text{let} \quad W = \frac{\Omega^2}{2} \qquad \text{then} \qquad \frac{\partial W}{\partial t} + \nabla\cdot(W\,\mathbf{v}) = 0$$

$$\text{let} \quad U = \frac{v^2}{2} \qquad \text{then} \qquad \frac{\partial U}{\partial t} + \nabla\cdot(U\,\mathbf{v}) = 0$$

both are conserved with same flow field

The Importance of Two Dimensionality

- in fluid dynamics, 2D can be forced by
  - strong rotation (Proudman-Taylor theorem)
  - domain anisotropy (the "thin atmosphere" situation)

- in plasma dynamics, 2D is usually forced by
  - strong background magnetic field ("guide field"), with Alfvén velocity v_A
  - specific energy density of reservoir $\ll v_A^2$
  - main reason: "low beta" meaning $T_e \ll M_i v_A^2$ hence $\beta_e = 4\pi p_e / B^2 \ll 1$

- in 2D, enstrophy is conserved; therefore

Kolmogorov cascade to small scales cannot occur in 2D

The Three Wave Interaction

- start with the 2D Euler equation

$$\frac{\partial \Omega}{\partial t} + \mathbf{v} \cdot \nabla \Omega = 0$$

- define Fourier decomposition

$$\psi = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \psi_{\mathbf{k}} \qquad \psi_{\mathbf{k}} = \oint \frac{k^2 d^2 x}{4\pi^2}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \psi \qquad \psi_{(-\mathbf{k})} = \psi^*_{\mathbf{k}}$$

- Euler equation in $\mathbf{k}$-space

$$\frac{\partial \Omega_{\mathbf{k}}}{\partial t} = \hat{\mathbf{s}} \cdot \oint \frac{k^2 d^2 x}{4\pi^2}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \sum_{-\mathbf{k}'} \sum_{-\mathbf{k}''} e^{-i\mathbf{k}'\cdot\mathbf{x}} e^{-i\mathbf{k}''\cdot\mathbf{x}} \,(i\mathbf{k}') \times (i\mathbf{k}'')\, \Omega_{-\mathbf{k}'} \psi_{-\mathbf{k}''}$$

- three wave condition for the integral not to vanish

$$\mathbf{k} + \mathbf{k}' + \mathbf{k}'' = 0$$

Equations for Beat Waves

- Euler equation

$$\frac{\partial \Omega_{\mathbf{k}}}{\partial t} = \sum_{-\mathbf{k}'} \sum_{-\mathbf{k}''} \frac{1}{2}\, \hat{\mathbf{s}} \cdot (\mathbf{k} \times \mathbf{k}')\, (\Omega_{-\mathbf{k}''} \psi_{-\mathbf{k}'} - \Omega_{-\mathbf{k}'} \psi_{-\mathbf{k}''})$$

- for beat waves use symmetry

$$\hat{\mathbf{s}} \cdot (\mathbf{k} \times \mathbf{k}') = \hat{\mathbf{s}} \cdot (\mathbf{k}' \times \mathbf{k}'') = \hat{\mathbf{s}} \cdot (\mathbf{k}'' \times \mathbf{k})$$

- define coupling matrix

$$C_{\mathbf{k}\mathbf{k}'} = \frac{1}{2}\, \hat{\mathbf{s}} \cdot (\mathbf{k} \times \mathbf{k}')$$

- find beat wave equations (use permutation among $\mathbf{k}, \mathbf{k}', \mathbf{k}''$ triangle)

$$\frac{\partial \Omega_{\mathbf{k}}}{\partial t} = C_{\mathbf{k}\mathbf{k}'}\, (\Omega_{-\mathbf{k}''} \psi_{-\mathbf{k}'} - \Omega_{\mathbf{k}'} \psi_{-\mathbf{k}''})$$

$$\frac{\partial \Omega_{\mathbf{k}'}}{\partial t} = C_{\mathbf{k}\mathbf{k}'}\, (\Omega_{-\mathbf{k}} \psi_{-\mathbf{k}''} - \Omega_{\mathbf{k}''} \psi_{-\mathbf{k}})$$

$$\frac{\partial \Omega_{\mathbf{k}''}}{\partial t} = C_{\mathbf{k}\mathbf{k}'}\, (\Omega_{-\mathbf{k}'} \psi_{-\mathbf{k}} - \Omega_{\mathbf{k}} \psi_{-\mathbf{k}'})$$

Energy Transfer

- find energy transfer by multiplying by $-\psi_{\mathbf{k}}$ and adding complex conjugate

$$\frac{\partial U_{\mathbf{k}}}{\partial t} = 2C_{\mathbf{k}\mathbf{k}'}\,\mathrm{Re}\left[\psi_{\mathbf{k}}\Omega_{\mathbf{k}'}\psi_{\mathbf{k}''} - \psi_{\mathbf{k}}\psi_{\mathbf{k}'}\Omega_{\mathbf{k}''}\right]$$

$$\frac{\partial U_{\mathbf{k}'}}{\partial t} = 2C_{\mathbf{k}\mathbf{k}'}\,\mathrm{Re}\left[\psi_{\mathbf{k}'}\Omega_{\mathbf{k}''}\psi_{\mathbf{k}} - \psi_{\mathbf{k}'}\psi_{\mathbf{k}''}\Omega_{\mathbf{k}}\right]$$

$$\frac{\partial U_{\mathbf{k}''}}{\partial t} = 2C_{\mathbf{k}\mathbf{k}'}\,\mathrm{Re}\left[\psi_{\mathbf{k}''}\Omega_{\mathbf{k}}\psi_{\mathbf{k}'} - \psi_{\mathbf{k}''}\psi_{\mathbf{k}}\Omega_{\mathbf{k}'}\right]$$

- identify transfer channel as terms with opposite sign in one pair of equations, e.g.,

$$T_U(\mathbf{k} \leftarrow \mathbf{k}') = 2C_{\mathbf{k}\mathbf{k}'}\,\mathrm{Re}\left[-\psi_{\mathbf{k}}\psi_{\mathbf{k}'}\Omega_{\mathbf{k}''}\right]$$

Enstrophy Transfer

- find enstrophy transfer by multiplying by $\Omega_{\mathbf{k}}$ and adding complex conjugate

$$\frac{\partial W_{\mathbf{k}}}{\partial t} = 2C_{\mathbf{k}\mathbf{k}'}\,\mathrm{Re}\left[\Omega_{\mathbf{k}}\psi_{\mathbf{k}'}\Omega_{\mathbf{k}''} - \Omega_{\mathbf{k}}\Omega_{\mathbf{k}'}\psi_{\mathbf{k}''}\right]$$

$$\frac{\partial W_{\mathbf{k}'}}{\partial t} = 2C_{\mathbf{k}\mathbf{k}'}\,\mathrm{Re}\left[\Omega_{\mathbf{k}'}\psi_{\mathbf{k}''}\Omega_{\mathbf{k}} - \Omega_{\mathbf{k}'}\Omega_{\mathbf{k}''}\psi_{\mathbf{k}}\right]$$

$$\frac{\partial W_{\mathbf{k}''}}{\partial t} = 2C_{\mathbf{k}\mathbf{k}'}\,\mathrm{Re}\left[\Omega_{\mathbf{k}''}\psi_{\mathbf{k}}\Omega_{\mathbf{k}'} - \Omega_{\mathbf{k}''}\Omega_{\mathbf{k}}\psi_{\mathbf{k}'}\right]$$

- identify transfer channel as terms with opposite sign in one pair of equations, e.g.,

$$T_W(\mathbf{k} \leftarrow \mathbf{k}') = 2C_{\mathbf{k}\mathbf{k}'}\,\mathrm{Re}\left[-\Omega_{\mathbf{k}}\Omega_{\mathbf{k}'}\psi_{\mathbf{k}''}\right]$$

The Dual Cascade

- write energy and enstrophy transfer

$$T_U(\mathbf{k} \leftarrow \mathbf{k}') = 2C_{\mathbf{k}\mathbf{k}'}\,\mathrm{Re}\left[-\psi_{\mathbf{k}}\psi_{\mathbf{k}'}\Omega_{\mathbf{k}''}\right] \qquad = 2C_{\mathbf{k}\mathbf{k}'}\,\mathrm{Re}\left[(k'')^2\psi_{\mathbf{k}}\psi_{\mathbf{k}'}\psi_{\mathbf{k}''}\right]$$

$$T_W(\mathbf{k} \leftarrow \mathbf{k}') = 2C_{\mathbf{k}\mathbf{k}'}\,\mathrm{Re}\left[-\Omega_{\mathbf{k}}\Omega_{\mathbf{k}'}\psi_{\mathbf{k}''}\right] \qquad = 2C_{\mathbf{k}\mathbf{k}'}\,\mathrm{Re}\left[-k^2(k')^2\psi_{\mathbf{k}}\psi_{\mathbf{k}'}\psi_{\mathbf{k}''}\right]$$

- note that given a definite sign of the triple correlation $[\psi_{\mathbf{k}}\psi_{\mathbf{k}'}\psi_{\mathbf{k}''}]$, these are opposite!
- statistically, enstrophy goes to higher k, hence smaller scale, due to the larger k-dependence

  † faster mixing, spectral redistribution

- hence energy goes preferentially to lower k, *hence larger scale*

2D inverse energy cascade

- "maximum entropy" stationary states for discrete systems show $W_k \sim k$ and $U_k \sim k^{-1}$

A Passive Scalar

- density fluctuations follow incompressible equation

$$\frac{\partial\widetilde{\rho}}{\partial t} + \mathbf{v} \cdot \nabla\widetilde{\rho} = 0$$

- passive scalar: $\widetilde{\rho}$ is advected by the flow, but effects no back reaction
- in $\mathbf{k}$-space the density equation is the same as for the vorticity
- "fluctuation free energy" or "entropy" is defined by squared amplitude
- hence the free energy transfer has the same form as for enstrophy

flow energy to large scales, free energy to small

- very high correlation $\widetilde{\Omega} \leftrightarrow \widetilde{\rho}$ in forced/dissipative turbulence, even with no coupling effects

Incompressible MHD

- constant parameters, homogeneous background, keep only quadratic nonlinearities

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla \left(p + \frac{B^2}{8\pi} \right) + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi}$$

- set $\mathbf{B} = B\mathbf{b}$ and $\mathbf{u} = \mathbf{v}/v_A$ with $v_A^2 = B^2/4\pi\rho$

$$\frac{1}{v_A} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{B^2} \nabla \left(4\pi p + \frac{B^2}{2} \right) + \mathbf{b} \cdot \nabla \mathbf{b}$$

- incompressible MHD kinematic equation

$$\frac{1}{v_A} \frac{\partial \mathbf{b}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{u}$$

- define “Elsässer variables” $\mathbf{u}_\pm = \mathbf{u} \pm \mathbf{b}$
- find passive advection, but coupled through advector (note $\nabla \cdot \mathbf{v} = 0 \leftrightarrow B^2$ not p, for $\beta \ll 1$)

$$\frac{1}{v_A} \frac{\partial \mathbf{u}_\pm}{\partial t} + \mathbf{u}_\mp \cdot \nabla \mathbf{u}_\pm = -\frac{1}{B^2} \nabla \left(4\pi p + \frac{B^2}{2} \right)$$

2D Incompressible MHD

- constant parameters, homogeneous background, keep only quadratic nonlinearities

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)=-\nabla\left(p+\frac{B^2}{8\pi}\right)+\frac{\mathbf{B}\cdot\nabla\mathbf{B}}{4\pi}$$

- take curl, use 2D to avoid $(\nabla\times\mathbf{v})\cdot\mathbf{v}$ and $\mathbf{J}\cdot\nabla\mathbf{B}$

$$\rho\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\nabla\times\mathbf{v}=\frac{1}{c}\mathbf{B}\cdot\nabla\mathbf{J}$$

- define ExB velocity and vorticity, parallel current, parallel gradient

$$\mathbf{v}=\mathbf{v}_E=\frac{c}{B^2}\mathbf{B}\times\nabla\phi \qquad \Omega=\frac{\rho c^2}{B^2}\nabla_\perp^2\phi \qquad J_\parallel=\mathbf{b}\cdot\mathbf{J}$$

- find correction to Euler vorticity equation

$$\frac{\partial\Omega}{\partial t}+\mathbf{v}_E\cdot\nabla\Omega=\mathbf{b}\cdot\nabla J_\parallel$$

applications of 2D incompressible MHD

- usually formulated with Elsässer variables: $\mathbf{u}_\pm = \mathbf{u} \pm \mathbf{b}$
- define velocity and magnetic field

$$\mathbf{u} = \hat{\mathbf{s}}\times\nabla\phi \qquad\qquad \mathbf{b} = -\hat{\mathbf{s}}\times\nabla\psi$$

- resistive (η), viscous (μ) MHD equations in Alfvén normalisation ($\partial/\partial t \leftrightarrow v_A\nabla$)

$$\frac{\partial \mathbf{u}_\pm}{\partial t} + \mathbf{u}_\mp \cdot \nabla \mathbf{u}_\pm = -\nabla I + (\mu \pm \eta)\nabla_\perp^2 \mathbf{u}_\pm$$

- incompressibility potential

$$\nabla^2 I + \nabla \cdot (\mathbf{u}_\mp \cdot \nabla \mathbf{u}_\pm) = 0$$

- this dynamical system is commonly used in astrophysics (e.g., reconnection, dynamo)
- for turbulence within an MHD stable equilibrium, the drive source is ∇p

  † coupling processes specifically in the electrons $p_e \leftrightarrow \phi$ become significant

  † and the MHD model cannot cover the physics . . .

Dissipative Coupling

- beyond MHD, density is not passive, but coupled through parallel currents to the ExB vorticity
- Ohm's law, parallel, keeping electron pressure gradient

$$-E_{\|} = \nabla_{\|}\widetilde{\phi} = \frac{1}{n_e e}\nabla_{\|}\widetilde{p}_e - \eta_{\|}\widetilde{J}_{\|}$$

- parallel compressibility enters electron pressure equation (advection is by the ExB velocity)

$$\frac{\partial \widetilde{p}_e}{\partial t} + \mathbf{v}_E \cdot \nabla \left(p_e + \widetilde{p}_e\right) = \frac{T_e}{e}\nabla_{\|}\widetilde{J}_{\|}$$

- appears as parallel diffusivity but couples to $\widetilde{\phi}$

$$\frac{\partial \widetilde{p}_e}{\partial t} + \mathbf{v}_E \cdot \nabla \left(p_e + \widetilde{p}_e\right) = \frac{T_e}{n_e e^2 \eta_{\|}}\nabla_{\|}^2 \left(\widetilde{p}_e - n_e e \widetilde{\phi}\right)$$

- note that $\nabla_{\|}\widetilde{p}_e \sim n_e e \nabla_{\|}\widetilde{\phi}$ is the *usual situation* in gradient driven turbulence

  † it cannot be treated by the single fluid MHD model

Dissipative Coupling Model for ExB Turbulence

- electrostatic approximation for $\omega \ll k_\perp v_A$

$$\mathbf{E}_\perp = -\nabla_\perp \phi$$

- electrostatic potential is stream function for ExB velocity

$$\mathbf{v}_E = \frac{c}{B^2}\mathbf{B}\times\nabla\phi \qquad\qquad \nabla\times\frac{\rho c}{B}\mathbf{v}_E = \Omega\mathbf{b}$$

- vorticity equation is the same as for MHD, with parallel gradient reckoned against the background

$$\frac{\partial\Omega}{\partial t} + \mathbf{v}_E\cdot\nabla\Omega = \nabla_\parallel J_\parallel$$

- changes are in the dissipative Ohm's law . . . and in the electron pressure equation

$$\eta_\parallel J_\parallel = \frac{1}{n_e e}\nabla_\parallel p_e - \nabla_\parallel\phi \qquad\qquad \frac{\partial p_e}{\partial t} + \mathbf{v}_E\cdot\nabla p_e = \frac{T_e}{e}\nabla_\parallel J_\parallel$$

- with $J_\parallel$ as a function of p_e and ϕ, the system is closed

Dissipative Coupling Model, properly 2D

- with no magnetic fluctuations, $\nabla_\parallel$ is slightly cheating
- actual dynamics is 3D, perp incompressible, $J_\parallel$ dynamics along $\mathbf{B}$ to provide coupling
- answer: model $-\nabla_\parallel^2$ with a positive coupling constant, with units of frequency

$$D = \frac{T_e}{n_e e^2 \eta_\parallel} k_\parallel^2 \qquad \text{which with} \quad \eta_\parallel = 0.51 \frac{m_e \nu_e}{n_e e^2} \qquad \text{becomes} \quad D = \frac{V_e^2}{0.51 \nu_e} k_\parallel^2$$

where $V_e = \sqrt{T_e/m_e}$ is the electron thermal velocity

- scale fluctuations as $e\widetilde{\phi}/T_e$ and $\widetilde{p}_e/p_e$, use $\rho = n_i M_i$ and $n_i = n_e$
- resulting model is called "Hasegawa-Wakatani"

$$\frac{c^2 M_i T_e}{e^2 B^2} \left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) \nabla_\perp^2 \frac{e\widetilde{\phi}}{T_e} = D \left(\frac{\widetilde{p}_e}{p_e} - \frac{e\widetilde{\phi}}{T_e} \right)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) \frac{\widetilde{p}_e}{p_e} + \mathbf{v}_E \cdot \nabla \log p_e = D \left(\frac{\widetilde{p}_e}{p_e} - \frac{e\widetilde{\phi}}{T_e} \right)$$

Dissipative Coupling Model, notes

- we've used a static, resistive, current

  † neglects magnetic induction $\leftrightarrow$ effects of $\partial\mathbf{B}/\partial t$, fails if $\omega \sim k_\parallel v_A$

- we've still used the ExB velocity for both ions and electrons, perp to $\mathbf{B}$

  † for MHD the only restriction is that electrostatic form requires $\omega \ll k_\perp v_A$

  † in general we require cold ions to use the ExB inertia term

$$n_e e \nabla_\perp \phi \sim \nabla_\perp p_e \ll \nabla_\perp p_i \qquad \text{requires} \quad T_i \ll T_e$$

- we've assumed isothermal electrons in the $\widetilde{p}_e$ equation

  † constant mass density is still OK if $\widetilde{p}_e \ll p_e$

  † generally, $\widetilde{T}_e$ is required but adds no qualitative changes, hence neglected in simplest model

- use of cold ions allows neglect of finite gyroradius effects and still reach down to drift scale

- we've neglected sound wave effects, reasonable if $k_\parallel L_\perp \ll 1$

- note that to *compare NUMBERS to an experiment* requires absolute complexity

Scales in the Dissipative Coupling Model

- the Hasegawa-Wakatani equations: dissipative coupling and gradient forcing

$$\frac{c^2 M_i T_e}{e^2 B^2}\left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla\right) \nabla_\perp^2 \frac{e\widetilde{\phi}}{T_e} = D\left(\frac{\widetilde{p}_e}{p_e} - \frac{e\widetilde{\phi}}{T_e}\right)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla\right) \frac{\widetilde{p}_e}{p_e} + \mathbf{v}_E \cdot \nabla \log p_e = D\left(\frac{\widetilde{p}_e}{p_e} - \frac{e\widetilde{\phi}}{T_e}\right)$$

- introduces the drift scale ρ_s, defined by

$$\rho_s^2 = c^2 M_i T_e / e^2 B^2$$

- gradient forcing gives the time scale $L_\perp / c_s$, from the sound speed c_s and profile scale length $L_\perp$

$$c_s^2 = \frac{T_e}{M_i} \qquad\qquad L_\perp = |\nabla \log p_e|^{-1}$$

- most interesting effects come from the varying properties of the two nonlinearities ...

Computational Dissipative Coupling Model

- normalise in terms of ρ_s and $c_s/L_\perp$, scale variables by a factor of $\delta = \rho_s/L_\perp$

$$\phi \leftarrow \delta^{-1}\, e\widetilde{\phi}/T_e \qquad p \leftarrow \delta^{-1}\, \widetilde{p}_e/p_e \qquad \Omega \leftarrow \delta^{-1}\, \rho_s^2 \nabla_\perp^2 \left(e\widetilde{\phi}/T_e \right)$$

- only parameter is $D \leftarrow DL_\perp/c_s$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) \Omega = D\,(p - \phi)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) p = -\frac{\partial \phi}{\partial y} + D\,(p - \phi)$$

- ExB advection defined in terms of a Poisson bracket structure, e.g.,

$$\mathbf{v}_E \cdot \nabla p = [\phi, p] = \frac{\partial \phi}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \phi}{\partial y}\frac{\partial p}{\partial x}$$

- linear forcing terms are the dissipative coupling (D) and the gradient drive: $v_E^x = -\partial\phi/\partial y$

Illustration of Dual Cascade

- periodic domain, $(20\pi\,\rho_s)^2$
- examine decaying turbulence started in middle of spectrum (set gradient drive to zero)

$$p_{\mathbf{k}}(0) = \phi_{\mathbf{k}}(0) = a_0 \left[1 + (k_\perp^2/0.32)^4\right]^{-1/2} e^{i\Theta}$$

  † random phase Θ

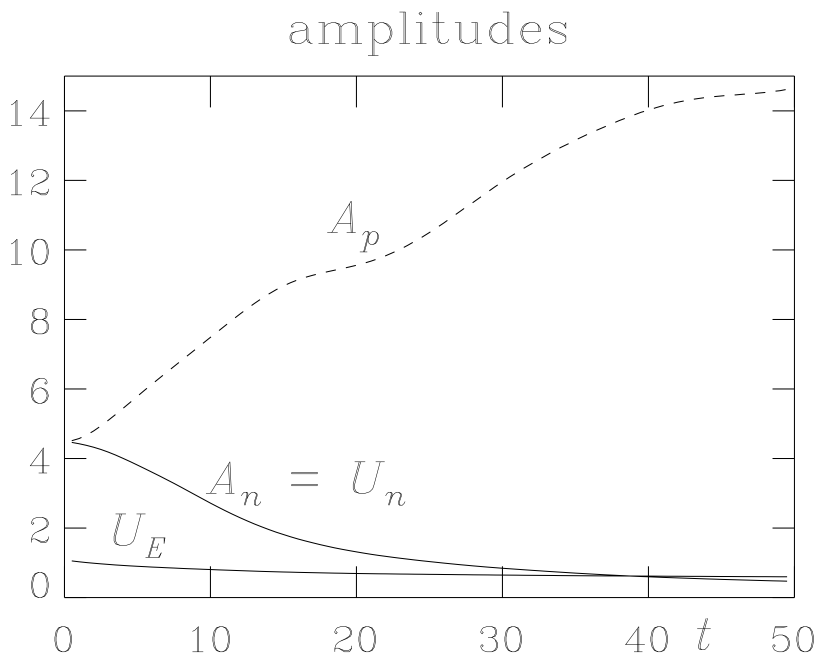
  † a_0 chosen such that rms amplitude is 3.0

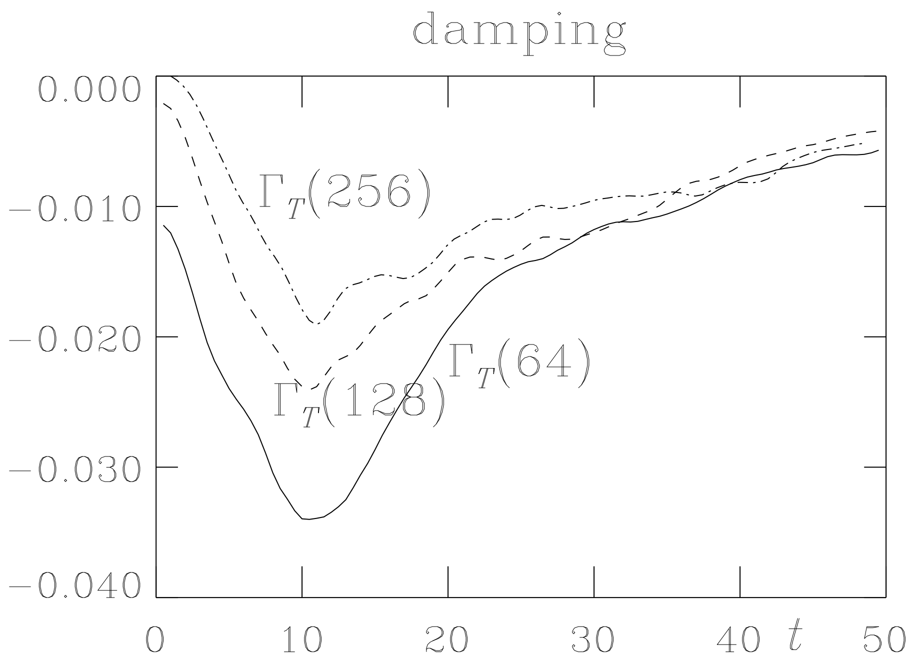
- test “hydrodynamic” limit $D = 0$

  † Euler equation for Ω, passive advection for p

- note in some of the figures label for p is n_e

- Time evolution of the hydrodynamic model



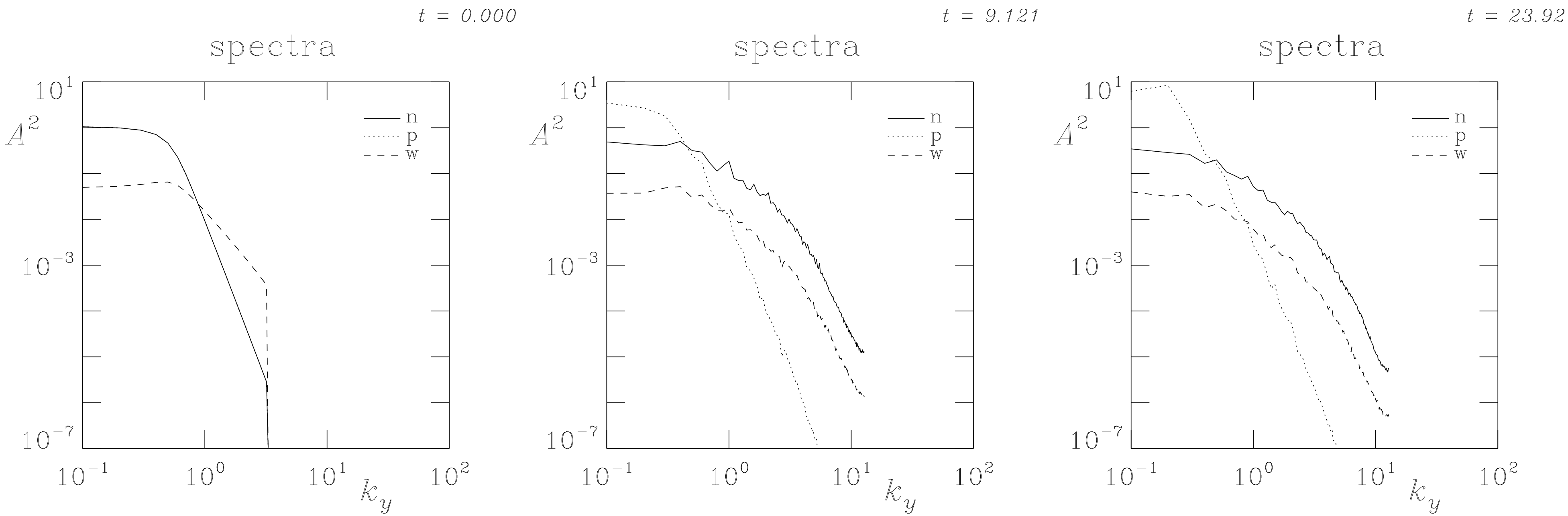


- initial decay of half squared amplitudes of p and ϕ, denoted A_n and A_p, respectively

  † also ExB energy (U_E) and fluctuation free energy (U_n)

- energetic losses (mostly in p due to the direct cascade) for three values of the resolution

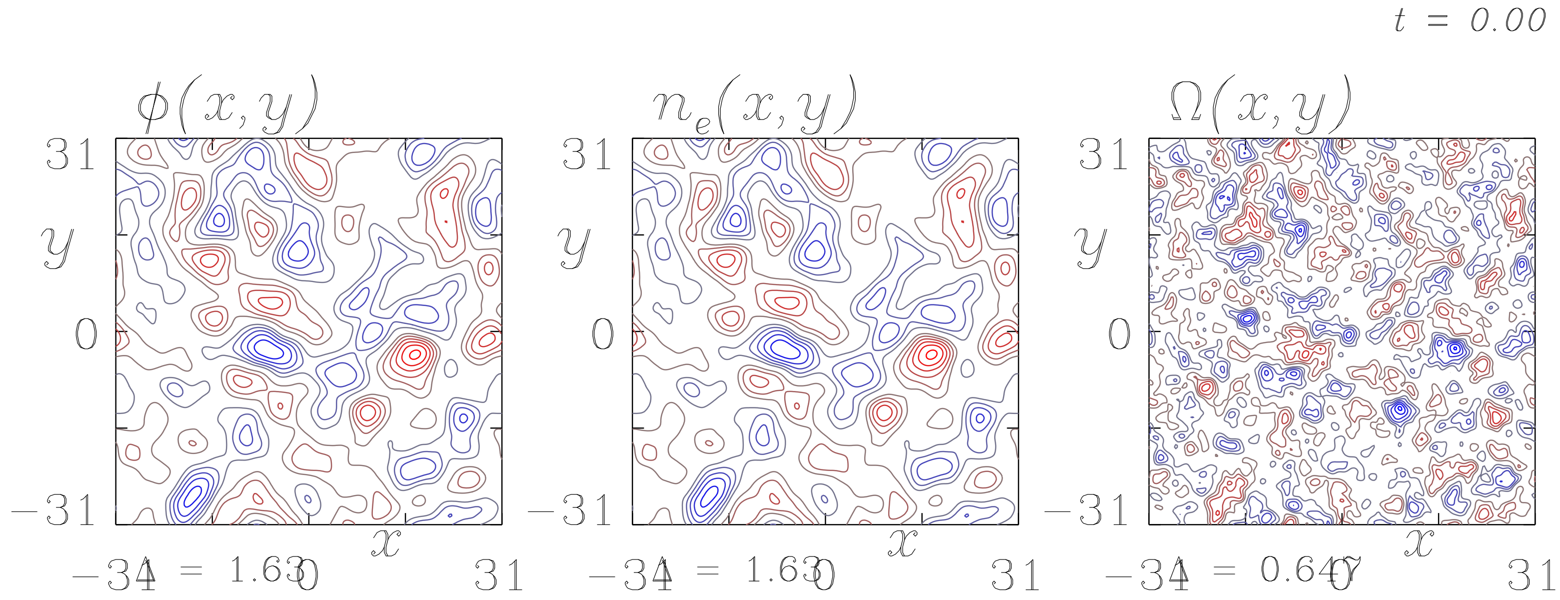
- Amplitude spectra in the hydrodynamic model, for p, ϕ, and Ω ('n', 'p', and 'w')



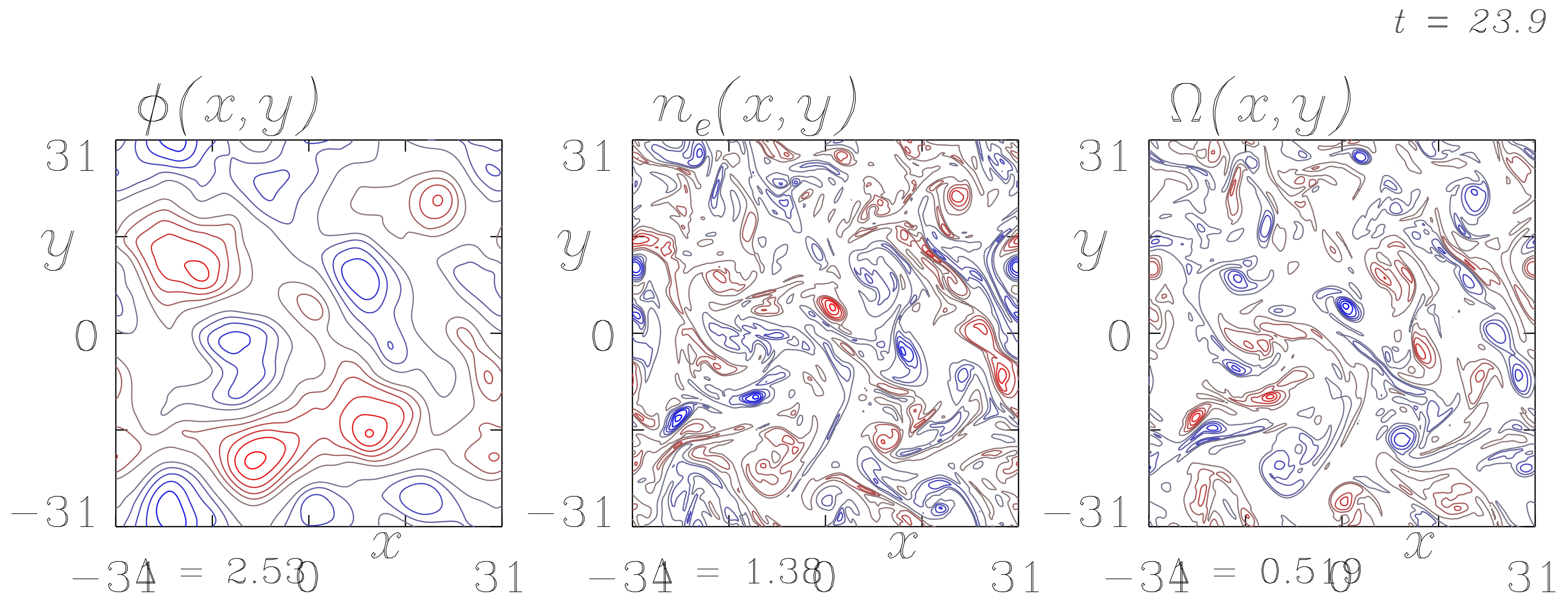
- times of the snapshots are $t = 0$ (left), $t = 9.8$ (center), and $t = 24$ (right)
- the spectra evolve rapidly apart due to the differing cascade dynamics for p and Ω versus ϕ

demonstration of bi-directional spectral transfer blank page to fool ps2pdf

- Evolution of the disturbances for the hydrodynamic model (note $n_e = p$)



- note that the morphology of Ω and ϕ is completely different although $\Omega = \nabla_\perp^2 \phi$



Energy Transfer in the Dissipative Coupling Model

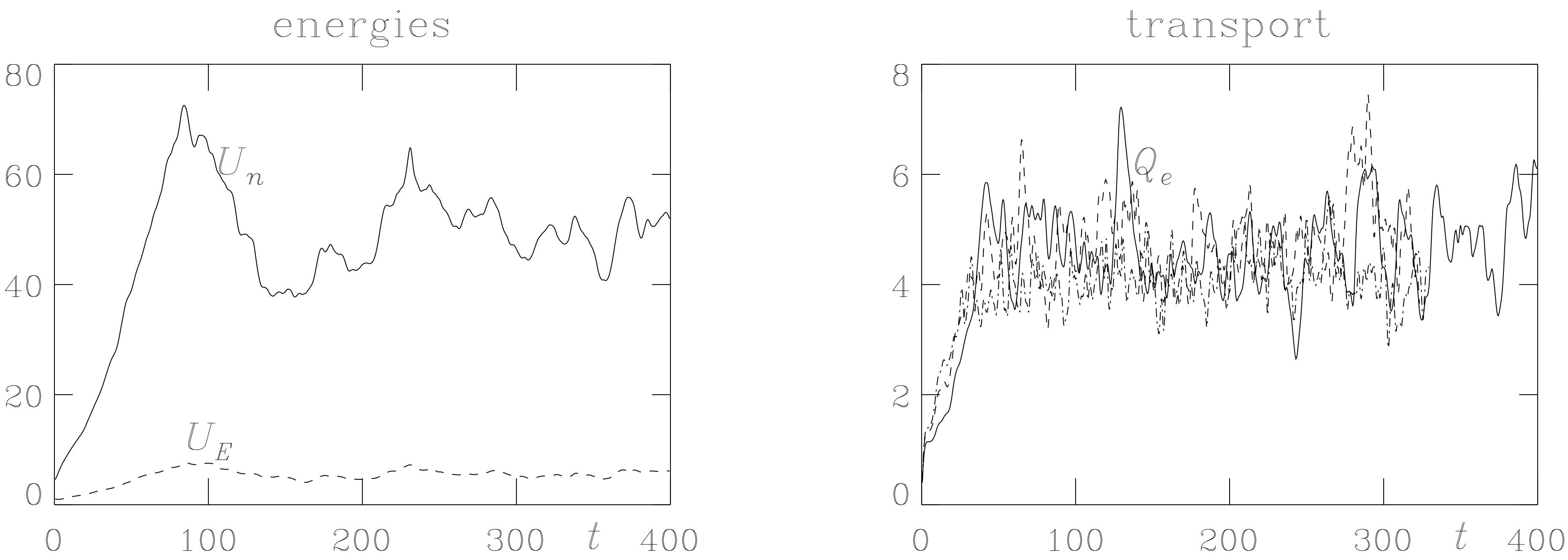
- switch gradient drive back on
- run to "saturation" defined by statistical stationarity for spectral quantities
- wide range of coupling strength, $D = 0.01$, 0.03, 0.1, 0.3, and 1.0
- displayed for $D = 0.1$

  † energy transfer directions hold for all D checked, only the robustness changes

- $D \to \infty$ is the "adiabatic limit" where $J_{\|} \to 0$ and $\phi \to p$

  † robustness of $\mathbf{v}_E \cdot \nabla p$ proportional to about $D^{-3/4}$

- Time evolution of the dissipative coupling model, for the nominal case of $D = 0.1$



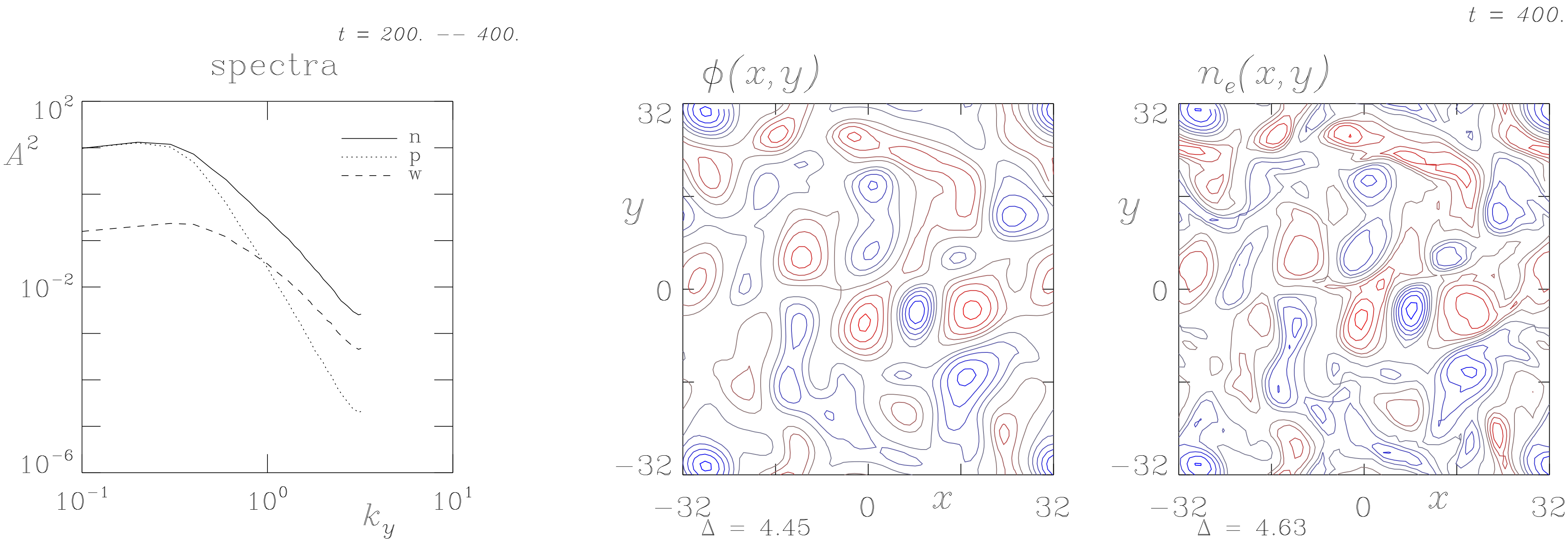
- half squared amplitudes of p and ϕ, denoted A_n and A_p, respectively

  † also ExB energy (U_E) and fluctuation free energy (U_n)

- transport caused by the turbulence, $Q_e = \langle p v_E^x \rangle$, for three values of the resolution

  † for 64^2, 128^2, and 256^2 grid nodes, the values are 4.69 ± 0.80 and 4.89 ± 0.74 and 4.14 ± 0.51

- Saturated state of the dissipative coupling model, for the nominal case of $D = 0.1$

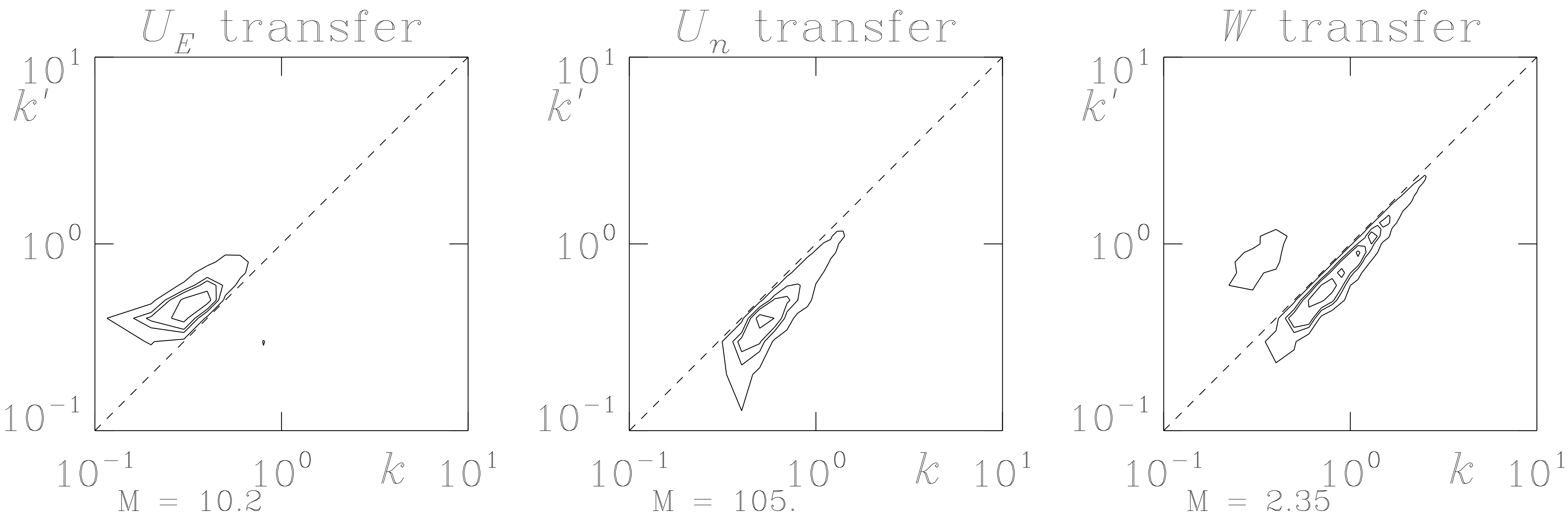


- averaged amplitude spectra for p, ϕ, and the Ω ('n', 'p', and 'w')
- morphology of ϕ and $n_e = p$ at $t = 400$

  † close coupling at larger scales but differences on smaller scales, corresponding to the spectra

  † the nonlinear interactions affecting p are stronger relative to the coupling at higher $k_\perp$

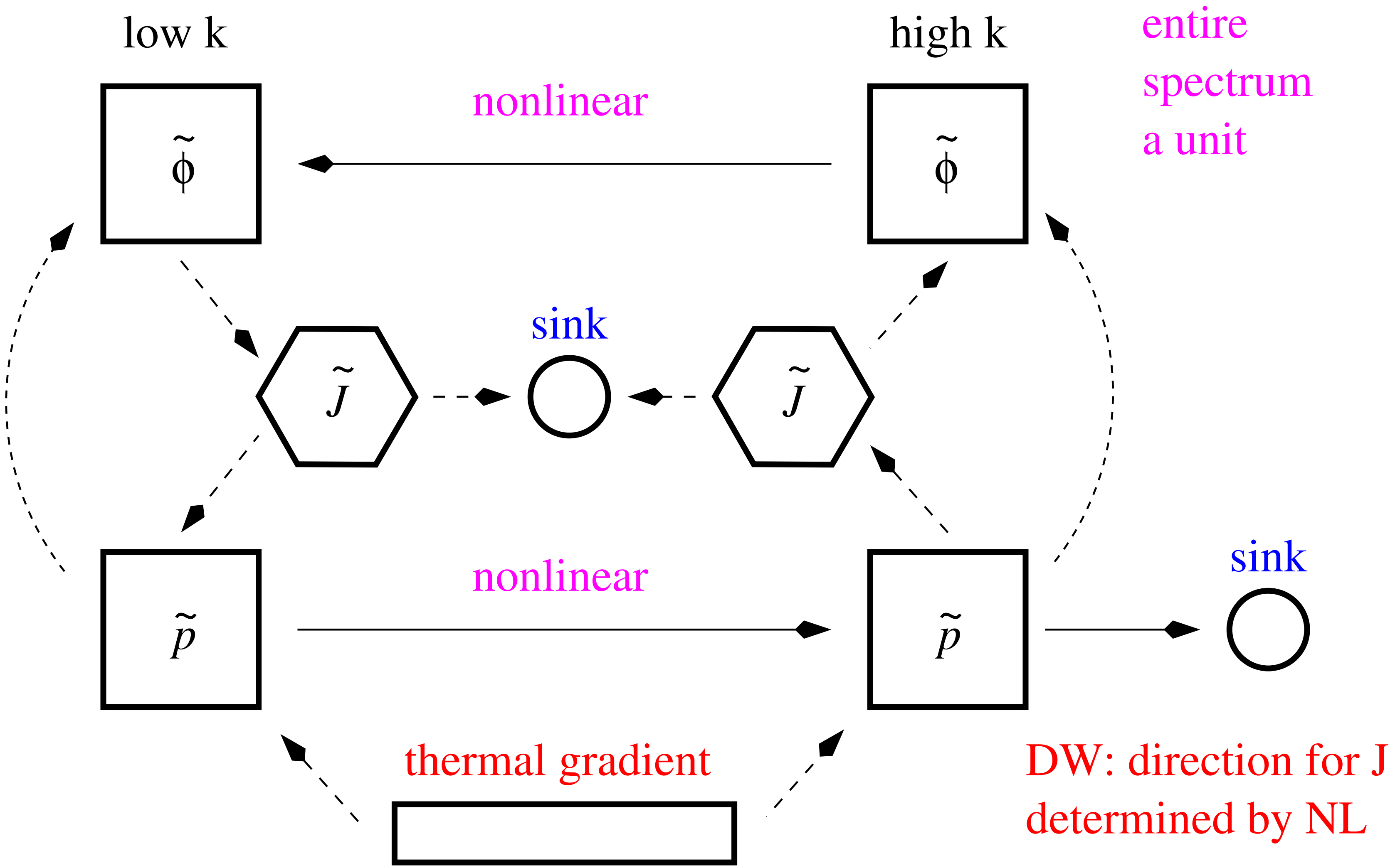
- Energy and enstrophy transfer in the dissipative coupling model, with $D = 0.1$



- transfer is from k' to k, shown where positive
- results show local cascade: mostly $1/2 < k'/k < 2$
- direct cascade for U_n and W, ... inverse cascade for U_E

cascade dynamics not changed by linear forcing

Energy Transfer: electromagnetic turbulence



(B Scott Phys Fluids B 1992, Plasma Phys Contr Fusion 1997)

(S Camargo et al Phys Plasmas 1995 and 1996)

Transport due to ExB Turbulence

- the turbulence causes a finite average advective transport, in general ...

$$Q = Q_e + Q_i \qquad Q_e = \left\langle \frac{3}{2}\,\widetilde{p}_e\, v_E^x \right\rangle \qquad Q_i = \left\langle \frac{3}{2}\,\widetilde{p}_i\, v_E^x \right\rangle$$

- in a confined plasma, the equilibrium is maintained by a source

$$\oint dV\, S = \oint dS\, Q = \oint dV\, \frac{\partial Q}{\partial x}$$

- the *time scales* are very different; typical values: $\delta \sim 10^{-2}$

$$\tau_{\text{turb}} \sim 200\, \frac{L_\perp}{c_s} \qquad \tau_{\text{source}} \sim \text{few} \times \delta^{-2}\, \frac{L_\perp}{c_s}$$

- profiles evolve slowly, turbulence in quasistatic *statistical* equilibrium

it is a good approximation to consider
turbulence in the presence of a prescribed gradient